

# Bichromatic compatible matchings

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**Abstract.** For a set  $R$  of  $n$  red points and a set  $B$  of  $n$  blue points, a *BR-matching* is a non-crossing geometric perfect matching where each segment has one endpoint in  $B$  and one in  $R$ . Two *BR-matchings* are compatible if their union is also non-crossing. We prove that, for any two distinct *BR-matchings*  $M$  and  $M'$ , there exists a sequence of *BR-matchings*  $M = M_1, \dots, M_k = M'$  such that  $M_{i-1}$  is compatible with  $M_i$ . This implies the connectivity of the *compatible bichromatic matching graph* containing one node for each bichromatic matching and an edge joining each pair of compatible matchings, thereby answering the open problem posed by Aichholzer et al. in [3].

## 1 Introduction

A planar straight line graph (PSLG) is a geometric graph in which the vertices are points embedded in the plane and the edges are non-crossing line segments. There are many special types of PSLG's of which we name a few. A triangulation is a PSLG to which no more edges may be added between existing vertices. A geometric matching of a given point set  $P$  is a 1-regular PSLG consisting of pairwise disjoint line segments in the plane joining points of  $P$ . Two PSLG's on the same vertex set are *compatible* if their union is planar.

Two branches of study on PSLG's include those of geometric augmentation and geometric reconfiguration. A typical augmentation problem on PSLG  $G = (V, E)$  asks for a set of new edges  $E'$  such that  $G' = (V, E \cup E')$  retains or gains some desired properties (see survey by Hurtado and Tóth [7]).

A typical reconfiguration problem on a pair of PSLG's  $G$  and  $G'$  sharing some property asks for a sequence of PSLG's  $G = G_0, \dots, G_k = G'$  where each successive pair of PSLG's  $G_{i-1}, G_i$  jointly satisfy some geometric constraints. In some situations, a bound on the value of  $k$  is studied as well.

One such solved problem is that of reconfiguring triangulations: given two triangulations  $T$  and  $T'$ , one can compute a sequence of triangulations  $T = T_0, \dots, T_k = T'$  such that  $T_{i-1}$  can be reconfigured to  $T_i$  by flipping one edge. Furthermore, bounds on the value of  $k$  are known:  $O(n^2)$  edge flips are always sufficient [6] and  $\Omega(n^2)$  edge flips are sometimes necessary [5].

Compatible geometric matchings have been the object of study in both augmentation and reconfiguration problems. For example, the *Disjoint Compatible*

*Matching Conjecture* [2] was recently solved in the affirmative [8]: every geometric matching  $M$  of  $2n$  segments on  $4n$  points can be augmented by  $2n$  additional segments to form a PLSG that is the union of simple polygons.

Given two matchings  $M$  and  $M'$  of a given point set, the reconfiguration problem asks for a *compatible sequence* of matchings  $M = M_0, \dots, M_k = M'$  such that  $M_{i-1}$  is compatible with  $M_i$  for all  $i \in \{1, \dots, k\}$ . Aichholzer et al. [2] proved that there is always a compatible sequence of  $O(\log n)$  matchings that reconfigures any given matching into a canonical matching  $M^*$ . Thus the *compatible matching graph*, that has one node for each matching and an edge between any two compatible matchings, is connected with diameter  $O(\log n)$ . Razen [10] proved that the distance between two nodes in this graph is sometimes  $\Omega(\log n / \log \log n)$ .

Given a bicolored point set in general position, a *bichromatic matching* (*BR-matching* for short) is a geometric matching in which each segment has one red and one blue endpoint. These segments are called *bichromatic segments*. At least one *BR-matching*  $M^*$  can always be produced by recursively applying ham-sandwich cuts (see Chapter 3 of [9]). Notice that the general position assumption is sometimes necessary to guarantee the existence of a *BR-matching*. However, not all *BR-matchings* can be produced using ham-sandwich cuts. Furthermore, some point sets admit only one *BR-matching*, which must be produced in this way.

Two *BR-matchings* are *connected* if one can be reconfigured into the other via a compatible sequence of *BR-matchings*.

We prove that all *BR-matchings* of a given point set are connected, by using  $M^*$  as a canonical form. We do this as follows. Consider the first ham-sandwich cut line  $\ell$  used to construct  $M^*$ . In Sections 3 and 4, we show how to reconfigure any given *BR-matching* via a compatible sequence, so that the last matching in the sequence contains no segment intersecting  $\ell$ . In Section 5, we use this result recursively, on every ham-sandwich cut used to generate  $M^*$ , to show that any given *BR-matching* is connected with  $M^*$ . This implies the connectivity of the *compatible bichromatic matching graph* which answers in the affirmative the question posed by Aichholzer et al. in [3]. Moreover, we show that the distance between two nodes is sometimes  $\Omega(n)$ . Finally, in Section ?? we characterize the point sets admitting only one *BR-matching*.

## 2 Preliminaries

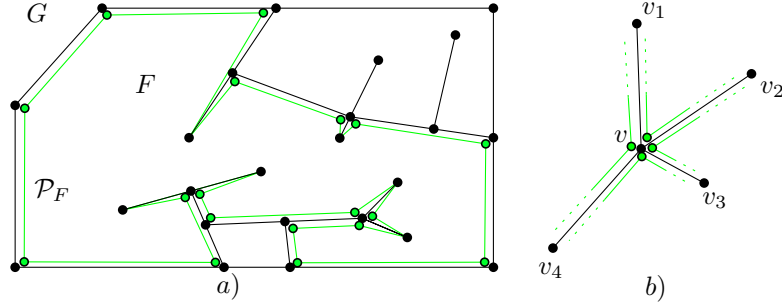
Let  $P = B \cup R$  be a set of points in the plane in general position where  $|R| = |B| = n$ . A straight-line segment with one endpoint in  $B$  and one in  $R$  is called a *bichromatic segment*. A point lies *above* a segment if it lies above the line extending that segment. A *bichromatic matching* of  $P$  is an independent set of bichromatic segments. Such a matching is perfect if every point of  $P$  belongs to exactly one segment and *planar* if no two segments cross. A bichromatic perfect planar matching is called a *BR-matching*. Two *BR-matchings*  $M$  and  $M'$  are

*compatible* if  $M \cup M'$  is planar and they are *connected* if there is a sequence of matchings  $M = M_0, \dots, M_r = M'$ , such that  $M_{i-1}$  is compatible with  $M_i$ ,  $1 \leq i \leq r$ .

We represent every bounded face  $F$  of a given PSLG as a sequence of its vertices in clockwise order along its boundary. We denote its interior by  $\text{int}(F)$  and its boundary by  $\partial F$ . A vertex  $v$  is *reflex* in  $F$  if there is a non-convex connected component in the intersection of  $\text{int}(F)$  with any disk centered at  $v$ . Notice that a vertex can be reflex in at most one face of a PSLG. A vertex of a PSLG is *reflex* if it is reflex in one of its bounded faces.

**Lemma 1.** *Every bounded face  $F$  of a PSLG contains a simple polygon  $\mathcal{P}_F$  (called the simplification of  $F$ ) such that  $F$  and  $\mathcal{P}_F$  have the same reflex vertices.*

*Proof.* Let  $F = (v_1, v_2, \dots, v_k, v_1)$  be a face of the given PSLG. For each vertex  $v_i$ , if the triple  $v_{i-1}, v_i, v_{i+1}$  makes a right turn, then let  $x_i$  be a point at distance  $\varepsilon > 0$  from  $v_i$ , lying on the bisector of the convex angle formed by  $[v_{i-1}, v_i]$  and  $[v_i, v_{i+1}]$ . If  $v_i$  is a reflex vertex, let  $x_i = v_i$ . Otherwise, if  $v_{i-1}, v_i, v_{i+1}$  are collinear, do nothing. Let  $\mathcal{P}_F = (x_1, \dots, x_k, x_1)$  (consider only the indices where  $x_i$  is defined). By choosing  $\varepsilon$  sufficiently small,  $\mathcal{P}_F$  is a simple polygon contained in  $F$  such that every reflex vertex  $v_j$  in  $F$  remains reflex in  $\mathcal{P}_F$  and no reflex vertex is created; see Fig. 1(a). Therefore,  $F$  and  $\mathcal{P}_F$  have the same set of reflex vertices.  $\square$



**Fig. 1.** a) A face  $F$  of a PSLG  $G$  and its simplification  $\mathcal{P}_F$ , contained in  $F$ , with the same set of reflex vertices. b) An isolated vertex  $v$  lying outside of the simplifications of each of its adjacent faces.

Let  $F_1, \dots, F_k$  be the bounded faces of a PSLG  $G$ . In the remainder, we will only consider the bounded faces when we refer to a face of a PSLG. The *boundary* of  $G$ , denoted by  $\partial G$ , is the union of all the edges in  $G$ . We call  $\text{int}(G) = \bigcup \text{int}(F_i)$  the *interior* of  $G$ . We call  $\mathcal{P}_G = \bigcup \mathcal{P}_{F_i}$  the *simplification* of  $G$ . Note that  $\mathcal{P}_G$  is the union of a set of disjoint simple polygons. A vertex  $v$  of  $G$  is *isolated* if no line through  $v$ , intersecting  $\text{int}(G)$ , supports a closed halfplane containing all the neighbors of  $v$ . An example of an isolated vertex is depicted in Fig. 1(b).

**Observation 1** *If  $v$  is an isolated vertex of  $G$ , then  $v$  lies outside of  $\mathcal{P}_G$ . Moreover, if  $v'$  is a vertex of  $G$  lying outside  $\mathcal{P}_G$ , then the open segments joining  $v'$  with its neighbors also lie outside of  $\mathcal{P}_G$ .*

Let  $M$  be a matching and let  $G$  be a PSLG containing all segments of  $M$  in its interior. We say that two points  $x$  and  $y$  in  $G$  (either in the interior or on the boundary) are *visible* if the open segment  $(x, y)$  lies in the interior of  $G$  and intersects no segment of  $M$ .

### 3 Well-colored graphs and basic tools

Let  $F$  be a face of a given PSLG with an even number of reflex vertices that are colored either blue or red. We say that  $F$  is *well-colored* if the sequence of reflex vertices along its boundary alternates in color. In the same way, a PSLG is well-colored if all its faces are well-colored.

#### 3.1 Coloring a PSLG

In this section, we define the color of a point, either on a bichromatic segment or on  $\partial G$ , depending on the position from which it is viewed; see Fig. 2(a) for an example.

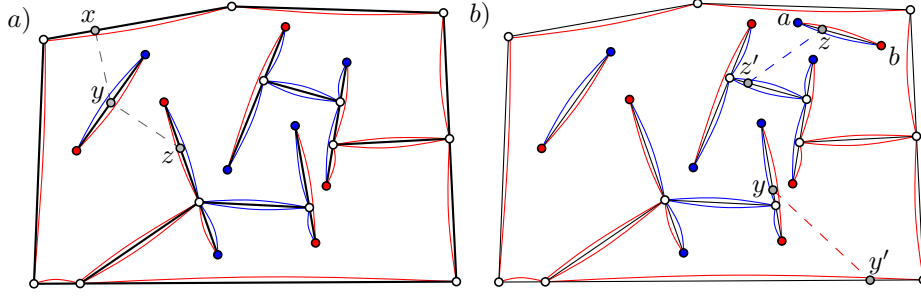
Assume that  $F$  is well-colored face of a PSLG and let  $x$  be a point on  $\partial F$ . Let  $y$  be a point in the plane such that  $x$  and  $y$  are visible. Walk in a straight line from  $y$  towards  $x$  and make a left turn when reaching  $x$ , following the boundary of  $F$  counterclockwise until reaching a reflex vertex  $r$  (if  $x$  is reflex, then  $r = x$ ). We say that  $x$  is blue (*resp.* red) when viewed from  $y$  if  $r$  is blue (*resp.* red). If  $F$  contains no reflex vertex, then the color of  $x$  when viewed from  $y$  can be arbitrarily blue or red.

This coloring scheme can be used for segments as well. For  $r \in R$  and  $b \in B$ , let  $x$  be a point in the interior of the bichromatic segment  $s = [r, b]$ ;  $x$  is blue when viewed from a point  $y$  in the plane not on the line extending  $s$  if the triple  $y, x, b$  makes a left turn; otherwise,  $x$  is red when viewed from  $y$ .

Let  $G$  be a well-colored PSLG and let  $M$  be a  $BR$ -matching with all its segments contained in the interior of  $G$ . Finally, let  $z$  and  $z'$  be two points such that each one lies either on  $\partial G$  or on a segment of  $M$ . We say that  $z$  and  $z'$  are *c-visible* if they are visible and the color of  $z$  when viewed from  $z'$  is equal to the color of  $z'$  when viewed from  $z$ .

#### 3.2 Basic operators for well-colored PSLG's

Let  $G$  be a well-colored PSLG and let  $M$  be a  $BR$ -matching contained in the interior of  $G$ . Let  $z$  be a point in the interior of a segment  $s = [a, b]$  of  $M$  and let  $z'$  be a point, but not a reflex vertex, on  $\partial G$  such that  $z$  and  $z'$  are c-visible. The operator  $\text{GLUE}(G, z, z')$  will attach  $s$  to  $\partial G$  using  $z$  and  $z'$  as points of attachment as follows: If  $z'$  is not a vertex of  $G$ , then insert it as a vertex by



**Fig. 2.** a) The coloring of the boundary points of a PSLG, as well as of a bichromatic segment. The point  $y$  is blue when viewed from  $x$  but red when viewed from  $z$ . Moreover,  $y$  and  $z$  are c-visible. b) Two pairs of c-visible points  $z, z'$  and  $y, y'$ , where  $z$  and  $z'$  can be joined by the GLUE operator and  $y$  and  $y'$  by the CUT operator.

splitting the edge of  $G$  that contains  $z'$ . Add the vertices  $z, a$  and  $b$  and the edges  $[z, z']$ ,  $[z, a]$  and  $[z, b]$  to  $G$ . Let  $\text{GLUE}(G, z, z')$  be the resulting PSLG where  $a$  and  $b$  are both reflex vertices of degree one; see Fig. 2(b).

Let  $y$  and  $y'$  be two c-visible points on  $\partial G$  such that neither  $y$  nor  $y'$  are reflex vertices. The operator  $\text{CUT}(G, y, y')$  is defined as follows: Let  $F$  be the face of  $G$  that contains the segment  $[y, y']$ . If either  $y$  or  $y'$  is not a vertex of  $G$ , insert it by splitting the edge where it belongs. Since  $[y, y']$  is a chord of  $F$ , adding the edge  $[y, y']$  to  $G$  forms two cycles and splits  $F$  into two new faces. In this way, we obtain a new PSLG  $\text{CUT}(G, y, y')$  with one face more than  $G$ ; see Fig. 2(b).

Since both operators join two points by adding the edge between them, we can define an operator  $\text{GLUECUT}(G, z, z')$ , that behaves like GLUE when  $z$  belongs to a segment in  $M$ , or behaves like CUT if both  $z$  and  $z'$  belong to  $\partial G$ .

A *Glue-Cut Graph (GCG)* is a well-colored PSLG where every reflex vertex has degree one.

**Lemma 2.** *The family of GCG's is closed under the GLUECUT operator.*

*Proof.* Let  $G$  be a GCG, let  $z$  be a point in a bichromatic segment  $s$  contained in the interior of  $G$  and let  $z', y$  and  $y'$  be points on  $\partial G$  such that  $z$  and  $z'$  (resp.  $y$  and  $y'$ ) are c-visible. When constructing  $\text{GLUE}(G, z, z')$ , the endpoints of  $s$  become reflex vertices of degree one. That is, we add one red and one blue reflex vertex to  $G$  hence, to prove that  $\text{GLUE}(G, z, z')$  is well-colored, it suffices to show that the points are added in the correct order with is guaranteed by the c-visibility of  $z$  and  $z'$ ; see Fig. 2(b).

On the other hand,  $\text{CUT}(G, y, y')$  neither adds nor removes reflex vertices of  $G$ . This operation divides a well-colored face of  $G$  into two, by inserting a new edge. Consider either of the new faces. Let  $a, b$  be the first reflex vertices found when following the boundary from this edge on each side. Since  $y$  and  $y'$  are

c-visible when CUT is invoked, we know that  $a$  and  $b$  are of different color. Thus each new face, and therefore  $\text{CUT}(G, y, y')$ , is well-colored; see Fig. 2(b).  $\square$

### 3.3 Merging a matching with a GCG

**Lemma 3.** (*Rephrasing of Lemma 5 of [1]*) *Let  $\mathcal{P}$  be a simple polygon. There exists a perfect planar matching  $M$  of the reflex vertices of  $\mathcal{P}$ , such that each segment of  $M$  is contained in (or on)  $\mathcal{P}$ .*

Let  $C = \{r_0, \dots, r_k\}$  be the set of reflex vertices of a simple polygon  $\mathcal{P}$  sorted along the boundary. Let  $M$  be the perfect planar matching of  $C$  which exists by Lemma 3. Let  $[r_i, r_j]$  be a segment of  $M$ , and note that this segment splits  $\mathcal{P}$  into two sub-polygons (note that one sub-polygon may be a segment). In order for  $M$  to be perfect and planar, each sub-polygon must contain an even number of reflex vertices. Therefore, if a segment  $[r_i, r_j]$  belongs to  $M$ , then  $i \bmod 2 \neq j \bmod 2$ . This implies that if  $\mathcal{P}$  is well-colored, then  $M$  is a *BR*-matching.

The main tool to construct *BR*-matchings of the reflex vertices of a GCG comes from the following lemma; see Fig. 3(a) for an illustration.

**Lemma 4.** *If  $G$  is a GCG, then there is a *BR*-matching  $M$  of the reflex vertices of  $G$ , such that each segment of  $M$  is contained in  $\mathcal{P}_G$ .*

*Proof.* Let  $F_1, \dots, F_k$  be the well-colored faces of  $G$ . By Lemma 1, each  $F_i$  and its simplification  $\mathcal{P}_{F_i}$  share the same set of reflex vertices. By Lemma 3, there is a matching  $M_i$  of the reflex vertices of  $\mathcal{P}_{F_i}$ , such that each segment lies either in the interior or on the boundary of  $\mathcal{P}_{F_i}$ . Since  $F_i$  is well-colored,  $M_i$  is a *BR*-matching. Note that a vertex can be reflex in at most one face of  $G$ . Therefore,  $M = \bigcup M_i$  is a *BR*-matching of the reflex vertices of  $G$  and each segment of  $M$  lies either in the interior or on the boundary of  $\mathcal{P}_G$ .  $\square$

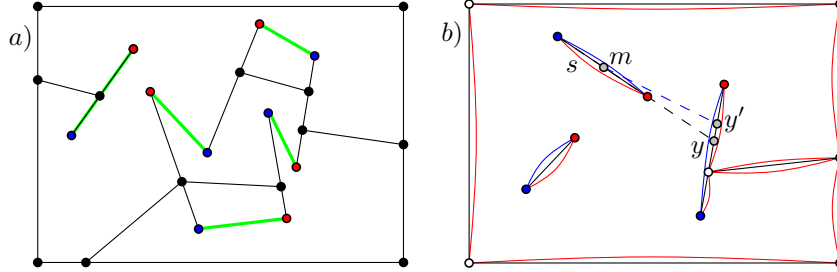
Let  $X$  be a GCG and let  $M$  be a *BR*-matching contained in the interior of  $X$ . In this section, we show how to glue the segments of  $M$  with the boundary of  $X$ , to obtain a GCG  $G$  such that the endpoints of the segments of  $M$  are all reflex vertices of  $G$ . Thus, by Lemma 4, we can obtain a *BR*-matching  $M'$  of the reflex vertices of  $G$  where every segment is contained in  $\mathcal{P}_G$ , i.e. we can obtain a *BR*-matching  $M'$  whose union with  $M$  contains no crossings.

Assume *wlog* that  $M$  contains no vertical segment. Also assume that the vertices of  $X$  and the endpoints of  $M$  are in general position.

Let  $s$  be the segment with the rightmost endpoint among all segments of  $M$ . We may assume that the left (*resp.* right) endpoint of  $s$  is blue (*resp.* red) hence  $s$  is blue (*resp.* red) when viewed from below (*resp.* above).

Extend  $s$  to the right until it intersects the interior of a segment  $s'$  on  $\partial X$  at a point  $y$ . Depending on the color of  $s'$  when viewed from  $s$ , choose a point  $y'$  in the interior of  $s'$  above (*resp.* below)  $y$  if  $s'$  is red (*resp.* blue). Choose  $y'$  sufficiently close to  $y$  so that the whole segment  $s$  is visible from  $y'$ . This is always possible since  $y$  is visible from the right endpoint of  $s$ . Let  $m$  be the midpoint of  $s$  and note that  $m$  and  $y'$  are c-visible by construction. Let  $X' = \text{GLUE}(X, m, y')$

and note that by Lemma 2,  $X'$  is a GCG. Moreover, the endpoints of  $s$  are reflex vertices of  $X'$ ; see Fig. 3(b). Remove  $s$  from  $M$ , let  $X = X'$  and repeat this construction recursively until  $M$  is empty. We obtain the following.



**Fig. 3.** a) A PSLG and a  $BR$ -matching of its reflex vertices. b) The gluing of a segment as described in Section 3.3.

**Lemma 5.** *Let  $X$  be a GCG and let  $M$  be a  $BR$ -matching contained in  $\text{int}(X)$ . There is a GCG  $G$  augmenting  $X$  such that all reflex vertices of  $X$  and all endpoints in  $M$  are reflex in  $G$ .*

## 4 Chromatic cuts and how to avoid them

Let  $M$  be a  $BR$ -matching of  $P$ . Given a line  $\ell$  that contains no endpoint of a segment in  $M$ , let  $S_{M,\ell}$  be the set of segments of  $M$  that properly cross  $\ell$ . We say that  $\ell$  is a *chromatic cut* of  $M$  if  $|S_{M,\ell}| \geq 2$  and not all endpoints of  $S_{M,\ell}$  on one side of  $\ell$  have the same color. Without loss of generality, we can assume that if a chromatic cut  $\ell$  exists, then it is vertical and no segment of  $M$  is parallel to  $\ell$ .

The objective of this section is to show that if a matching  $M$  and chromatic cut  $\ell$  are given, it is possible to obtain a new matching with at least one segment  $s$  of  $S_{M,\ell}$  absent. Furthermore, when examining segments that cross  $\ell$  “below”  $s$  on the new matching, all segments of  $S_{M,\ell}$  are preserved and no new segments are introduced.

### 4.1 Processing segments that intersect a chromatic cut

Let  $\ell$  be a chromatic cut of  $M$  and assume that  $S_{M,\ell} = \{s_1, \dots, s_k\}$  is sorted from bottom to top according to the intersection,  $x_i$ , of  $s_i$  with  $\ell$ .

**Lemma 6.** *There exist two consecutive segments  $s_i$  and  $s_{i+1}$  in  $S_{M,\ell}$ , such that  $x_i$  and  $x_{i+1}$  are  $c$ -visible, for  $1 \leq i \leq k-1$ .*

*Proof.* Since  $\ell$  is a vertical chromatic cut, there exist two segments  $s_j$  and  $s_h$  in  $S_{M,\ell}$  such that the left point of  $s_j$  is of different color than the left point of  $s_h$ . Therefore, two consecutive segments  $s_i$  and  $s_{i+1}$  must exist having left endpoints of different color. This implies that the color of  $s_i$  when viewed from above is the same as the color of  $s_{i+1}$  when viewed from below. Finally, since  $s_i$  and  $s_{i+1}$  are consecutive segments in  $S_{M,\ell}$ ,  $x_i$  and  $x_{i+1}$  are visible.  $\square$

Let  $\mathcal{R}$  be a sufficiently large convex polygon containing all segments of  $M$  in its interior. Assume *wlog* that the left endpoint of  $s_1$  is blue, implying that  $s_1$  is red from above and blue from below. Let  $x_0$  be the bottom intersection between  $\ell$  and  $\mathcal{R}$ . Since the bounded face of  $\mathcal{R}$  contains no reflex vertex, we can assume that  $x_0$  is blue when viewed from  $x_1$ . That is,  $x_0$  and  $x_1$  are c-visible. Finally, let  $X_1 = \text{GLUE}(\mathcal{R}, x_1, x_0)$  be the GCG obtained by joining  $x_0$  with  $x_1$ ; see Fig. 4(a).

If we consider the edge of  $\mathcal{R}$  containing  $x_0$  to be a segment  $s_0$ , then  $X_i$  is a GCG with the following properties, for  $i = 1$ .

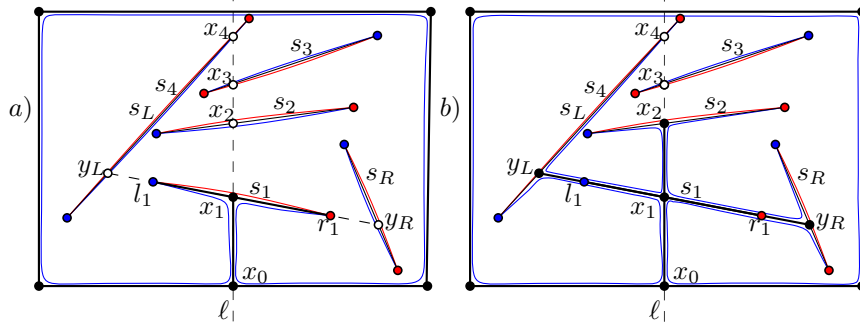
- The points  $x_i$  and  $x_{i+1}$  are visible while  $x_i$  and  $x_{i-1}$  are neighbors.
- In addition to  $x_{i-1}$ ,  $x_i$  neighbors two vertices on  $s_i$ , one to the left and one to the right of  $\ell$ .
- The endpoints of  $s_i$  are reflex vertices of  $X_i$ .
- The endpoints of  $s_{i-1}$  are not reflex and  $x_{i-1}$  lies outside of  $\mathcal{P}_{X_i}$ .
- The color of  $s_i$ , when viewed from a point lying above  $s_i$ , is given by the color of the right endpoint of  $s_i$ .

Our objective is to find a point in  $X_i$ , c-visible with  $x_i$ , that lies above the line extending  $s_i$ . As long as no such point exists, we iteratively augment  $X_i$  maintaining the above properties as an invariant. This is done with procedure **AUGMENT**( $i$ ), which takes a GCG  $X_i$  and adds edges (including the edge between  $x_i$  and  $x_{i+1}$ ) to produce a new GCG  $X_{i+1}$  where the above properties hold. The idea is that after several augmentations, we will produce a GCG where the desired c-visible point will be found.

**Procedure AUGMENT( $i$ )** Let  $l_i$  and  $r_i$  be the left and right endpoints of  $s_i$ , respectively. Assume *wlog* that  $l_i$  is colored blue (and  $r_i$  is red) hence  $s_i$  is red when viewed from above. Extend  $s_i$  on both sides and let  $s_L$  (*resp.*  $s_R$ ) be the first reached segment to the left (*resp.* right). This procedure is only used when the points in  $s_{i+1}$ ,  $s_L$  and  $s_R$  appear blue when viewed from  $x_i$ . Otherwise, there is a point in either  $s_{i+1}$ ,  $s_L$  or  $s_R$ , lying above  $s_i$ , that is c-visible with  $x_i$ .

Notice that  $s_{i+1}$ ,  $s_L$ , and  $s_R$  could belong either to  $M$ , or to  $\partial X_i$ . See Fig. 4(a). Let  $y_L$  and  $y_R$  be the points where the line extending  $s_i$  intersects  $s_L$  and  $s_R$ , respectively. Let  $X'_i$  be the PSLG obtained by adding the edges  $[l_i, y_L]$  and  $[r_i, y_R]$  to  $X_i$  ( $y_L$  and  $y_R$  are added as vertices). This may create new faces depending on whether  $s_L$  or  $s_R$  belong to  $M$ . Vertices  $y_L, l_i, x_i, r_i, y_R$  are collinear, meaning  $l_i$  and  $r_i$  are no longer reflex vertices in  $X'_i$ . Thus  $s_i$  will now be blue when viewed from both sides. Furthermore, if  $s_L$  or  $s_R$  belong to





**Fig. 4.** (a) Example where procedure AUGMENT(1) is required. Point  $x_1$  is not c-visible with any other edge; (b) The construction obtained by extending  $s_1$ , where two reflex vertices  $l_1, r_1$  disappear to let  $x_1$  and  $x_2$  become c-visible.

$M$ , then their endpoints are now reflex vertices of  $X'_i$ . One can verify that  $X'_i$  is well-colored since  $y_L$  and  $y_R$  are both blue when viewed from  $x_i$ , hence  $X'_i$  is a GCG. See Fig. 4. Notice that, when viewed from above, the color of  $x_i$  is now blue, in contrast with the red color that  $x_i$  had on  $X_i$ . Therefore, since  $x_{i+1}$  is blue when viewed from below,  $x_{i+1}$  and  $x_i$  are now c-visible in  $X'_i$ .

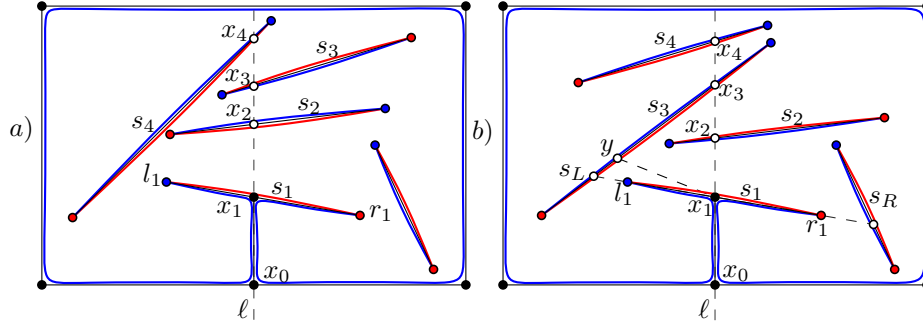
Let  $X_{i+1} = \text{GLUECUT}(X'_i, x_{i+1}, x_i)$ . This way, the endpoints of  $s_{i+1}$  become (if they were not already) reflex vertices of the GCG  $X_{i+1}$  and  $x_i$  becomes an isolated vertex. Thus, by Observation 1,  $x_i$  lies outside of  $\mathcal{P}_{X_{i+1}}$ . Notice that no vertex on  $s_{i+1}$  neighbors a point lying above  $s_{i+1}$ . Thus, the color of every point on  $s_{i+1}$ , when viewed from above, is given by the color of the right endpoint of  $s_{i+1}$ . In fact, the invariant properties are maintained, should there be a subsequent use of AUGMENT.

## 4.2 Analysis of AUGMENT

**Observation 2** *On each iteration of AUGMENT, all reflex vertices of  $X_i$  are preserved in  $X_{i+1}$ , except for the two endpoints of  $s_i$  that become non-reflex.*

**Lemma 7.** *The procedure AUGMENT will only be used  $O(n)$  times before producing a GCG  $X_j$ , where there exists a point, lying above the segment  $s_j$ , that is c-visible with  $x_j$  (for some  $1 \leq j \leq k-1$ ).*

*Proof.* By Lemma 6, there exist segments  $s_h, s_{h+1} \in S_{M,\ell}$  such that  $x_h$  and  $x_{h+1}$  are c-visible before executing AUGMENT on  $X_1$ . We claim that AUGMENT can only go as far as to construct  $X_h$ . If  $X_h$  is not constructed, it is because a GCG  $X_j$  was constructed (for some  $0 \leq j < h$ ), where there exists a point, lying above the segment  $s_j$ , that is c-visible with  $x_j$ . Otherwise, if  $X_h$  is constructed, then, by the preserved invariants, the color of  $x_h$ , when viewed from above, remains unchanged and hence  $x_h$  and  $x_{h+1}$  are c-visible.  $\square$



**Fig. 5.** (a) Example where  $x_1$  and  $x_2$  are c-visible; (b) Example where  $x_1$  and  $x_2$  are not c-visible, but point  $y$  can be found in  $s_L$  so that  $x_1$  and  $y$  are c-visible.

### 4.3 Processing after AUGMENT

From Lemma 7, we know that after the last call to AUGMENT we obtain a GCG  $X_j$  such that there is a point in either  $s_{j+1}$ ,  $s_L$  or  $s_R$ , lying above  $s_j$ , that is c-visible with  $x_j$ . Assume wlog that  $x_j$  is red when viewed from above.

If  $s_{j+1}$  is red when viewed from below, then  $x_j$  and  $x_{j+1}$  are c-visible. In this case we let  $G_{M,\ell} = \text{GLUECUT}(X_j, x_{j+1}, x_j)$ ; see Fig. 5(a).

Instead, if  $x_{j+1}$  is blue when viewed from  $x_j$ , we follow a different approach. Recall that the endpoints of  $s_i$  are reflex vertices. If  $s_L$  is red when viewed from the left endpoint of  $s_j$ , choose a point  $y$ , slightly above  $y_L$  on  $s_L$ , such that the whole segment  $s_j$  is visible from  $y$ . Since  $x_j$  is red when viewed from above,  $x_j$  and  $y$  are c-visible. Let  $G_{M,\ell} = \text{GLUECUT}(X_j, y, x_j)$ ; see Fig. 5(b). An analogous construction of  $G_{M,\ell}$  follows if  $s_R$  is red when viewed from the right endpoint of  $s_j$ . We call  $G_{M,\ell}$  the *extension* of  $X_j$ .

**Lemma 8.** *If  $G_{M,\ell}$  is an extension of  $X_j$ , then the following properties hold:*

- *The endpoints of  $s_j$  are reflex vertices of  $G_{M,\ell}$ , but  $s_j$  is not contained in  $\mathcal{P}_{G_{M,\ell}}$ .*
- *For every  $1 \leq h < j$ , the endpoints of  $s_h$  are not reflex vertices of  $G_{M,\ell}$ . Moreover,  $s_h$  is not contained in  $\mathcal{P}_{G_{M,\ell}}$ .*
- *The downwards ray with apex at  $x_j$  does not intersect  $\mathcal{P}_{G_{M,\ell}}$ .*

*Proof.* By the invariants of AUGMENT,  $x_j$  neighbors  $x_{j-1}$  as well as two vertices on  $s_j$ , one to the left and one to the right of  $\ell$ . Since  $x_j$  also neighbors a vertex in  $G_{M,\ell}$  lying above the segment  $s_j$ ,  $x_j$  is an isolated vertex in  $G_{M,\ell}$ . Thus, by the preserved invariants and by Observation 1, for every  $1 \leq i < j$ ,  $x_i$  lies outside of  $\mathcal{P}_{G_{M,\ell}}$  and hence the segment  $s_i$  is not contained in  $\mathcal{P}_{G_{M,\ell}}$ . Furthermore, the segment joining  $x_i$  with  $x_{i-1}$  also lies outside of  $\mathcal{P}_{G_{M,\ell}}$  and so does the downwards ray with apex at  $x_j$ . Finally, Observation 2 tells us that, for every  $1 \leq h < j$ , no endpoint of  $s_h$  is a reflex vertex of  $X_j$  (nor of  $G_{M,\ell}$ ).  $\square$

**Lemma 9.** *Let  $M$  be a BR-matching of  $P$  and let  $\ell$  be a chromatic cut of  $M$ . There exists a BR-matching  $M'$  of  $P$ , compatible with  $M$ , such that:*

- There is a segment  $s$  in  $S_{M,\ell}$  that does not belong to  $M'$ .
- Let  $x = s \cap \ell$ . All segments of  $M$ , intersecting  $\ell$  below  $x$ , belong to  $M'$ . Moreover, these are the only segments of  $M'$  intersecting  $\ell$  below  $x$ .

*Proof.* Let  $G_{M,\ell}$  be the GCG obtained on  $M$  and  $\ell$  by the construction presented in this section. Lemma 8 states that there is a segment  $s_j \in S_{M,\ell}$ , such that its endpoints are reflex vertices of  $G_{M,\ell}$  but  $s_j$  is not contained in  $\mathcal{P}_{G_{M,\ell}}$ . Let  $W$  be the set of segments in  $M$  that are contained in the interior of  $G_{M,\ell}$  and let  $Z_\ell = \{s_1, \dots, s_{j-1}\}$  be the set of segments of  $S_{M,\ell}$  that intersect  $\ell$  below  $x_j$ . From Lemma 8 we know that  $Z_\ell \cap W = \emptyset$ .

By Lemma 5, since  $W$  is contained in  $\text{int}(G_{M,\ell})$ , we can augment  $G_{M,\ell}$  by gluing the segments of  $W$  to its boundary such that the endpoints of every segment in  $W$  become reflex vertices in  $G_{M,\ell}$ . Moreover, the reflex vertices of  $G_{M,\ell}$  are preserved.

From Lemma 4, there exists a  $BR$ -matching  $W'$  of the reflex vertices of  $G_{M,\ell}$  such that each segment in  $W'$  is contained in  $\mathcal{P}_{G_{M,\ell}}$ . Notice that the endpoints of  $s_j$  are re-matched in  $W'$ . However, since  $s_j$  is not contained in  $\mathcal{P}_{G_{M,\ell}}$ ,  $s_j$  does not belong to  $W'$ . Moreover, Lemma 8 implies that the ray, shooting downwards from  $x_j$ , lies outside  $\mathcal{P}_{G_{M,\ell}}$ . Thus, no segment in  $W'$  intersects  $\ell$  below  $x_j$ .

Let  $M' = W' \cup Z_\ell$  be a set of bichromatic segments. Every point in  $P$  is matched in  $M'$  since every point in  $P$  is either a reflex vertex of  $G_{M,\ell}$ , or an endpoint of a segment in  $Z_\ell$ . Lemma 8 implies that the endpoints of the segments in  $Z_\ell$  are not reflex vertices in  $G_{M,\ell}$ . Therefore,  $M'$  is a perfect  $BR$ -matching of  $P$ . Since  $W$  and  $W'$  are compatible,  $M$  and  $M'$  are compatible matchings.  $\square$

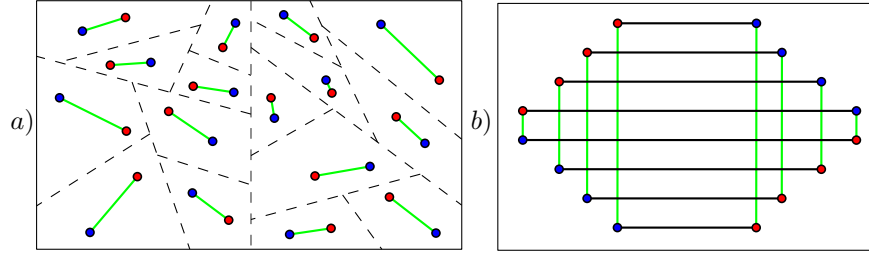
## 5 The ham-sandwich matching

In this paper, a *ham-sandwich cut* of  $P$  is a line passing through no point of  $P$  and containing exactly  $\lfloor \frac{n}{2} \rfloor$  blue and  $\lfloor \frac{n}{2} \rfloor$  red points to one side. Notice that if  $n$  is even, then our definition matches the *classical* definition of ham-sandwich cuts (see Chapter 3 of [9]). However, when  $n$  is odd, a ham-sandwich cut  $\ell$  according to the classical definition will go through a red and a blue point of  $P$ . In this case, we obtain a ham-sandwich cut, according to our definition, by slightly moving  $\ell$  away from these two points without changing its slope and without reaching another point of  $P$ . By the general position assumption this is always possible.

Since every bichromatic point set admits a ham-sandwich cut (see Chapter 3 of [9]),  $P$  admits at least one  $BR$ -matching resulting from recursively applying the ham-sandwich cut. We call this a *ham-sandwich matching*. However, notice that it is not necessarily unique; see Fig. 6.

**Observation 3** *Given any  $BR$ -matching  $M$  of  $P$ , every ham-sandwich cut of  $P$  is either disjoint from  $M$ , or is a chromatic cut of  $M$ .*

**Lemma 10.** *Given a  $BR$ -matching  $M$  of  $P$  and a ham-sandwich cut  $\ell$ , there is a matching  $M^\ell$  connected with  $M$  such that no segment of  $M^\ell$  intersects  $\ell$ .*



**Fig. 6.** a) A ham-sandwich matching obtained by recursive applications of the ham-sandwich cut. b) Two  $BR$ -matchings at distance  $\Omega(n)$  in the graph  $G_P$  that are both ham-sandwich matchings.

*Proof.* Assume that  $\ell$  is a chromatic cut of  $M$ . Otherwise, the result follows trivially. Given a  $BR$ -matching  $W$  of  $P$  such that  $\ell$  is a chromatic cut of  $W$ , let  $\text{NEXT}(W)$  be the matching, compatible to  $W$ , that exists as a consequence of Corollary 9. We claim that the sequence  $\varphi = (M_0, \dots, M_h = M^\ell)$  of compatible matchings defined as follows is finite:  $M_0 = M$ . If  $\ell$  is a chromatic cut of  $M_i$ , then  $M_{i+1} = \text{NEXT}(M_i)$ . Otherwise,  $M_h = M_i$ .

Observation 3 guarantees that  $\varphi$  is well defined. Assume *wlog* that  $\ell$  is a vertical line. Let  $\mathcal{C}_P = \{z_1, z_2, \dots, z_m\}$  be the set of all possible  $O(n^2)$  bichromatic segments that cross  $\ell$  with both endpoints in  $P$ . Assume that  $\mathcal{C}_P$  is sorted, from bottom to top, according to the intersection of each segment with  $\ell$ . Given a  $BR$ -matching  $W$  of  $P$ , let  $\chi_W = b_1 b_2 \dots b_m$  be a binary number where  $b_i$  is defined as follows:

$$b_i = \begin{cases} 1 & \text{If } z_i \text{ belongs to } M \\ 0 & \text{Otherwise} \end{cases}$$

Let  $M_i$  and  $M_{i+1}$  be two consecutive matchings in  $\varphi$ . By Corollary 9, there is a segment  $s$ , corresponding to a segment  $z_k$  in  $\mathcal{C}_P$ , such that  $s = z_k$  belongs to  $M_i$  but not to  $M_{i+1}$ . Moreover, if  $z_j$  is a segment that intersects  $\ell$  below  $z_k \cap \ell$ , then  $z_j$  belongs to  $M_i$  if and only if  $z_j$  belongs to  $M_{i+1}$ . Therefore, the  $k$ -th digit of  $\chi_{M_i}$  is 1 while the  $k$ -th digit of  $\chi_{M_{i+1}}$  is 0. Moreover, the  $j$ -th digit of  $\chi_{M_i}$  and  $\chi_{M_{i+1}}$  are equal for every  $j < k$ . This implies that  $\chi_{M_i} > \chi_{M_{i+1}}$ . Therefore,  $\Phi = \chi_{M_0}, \chi_{M_1}, \dots, \chi_{M_h}$  is a strictly decreasing sequence. This means that no  $BR$ -matching is repeated.  $\square$

**Theorem 4.** Let  $M$  be a  $BR$ -matching of  $P$  and let  $H$  be a ham-sandwich matching of  $P$ . There is a sequence of matchings  $M = M_0, \dots, M_r = H$ , such that  $M_i$  is compatible with  $M_{i+1}$  for  $0 \leq i \leq r-1$ .

*Proof.* Let  $\ell$  be the first ham-sandwich cut line used to construct  $H$ . By Lemma 10, there is a matching  $M^\ell$  such that  $M$  and  $M^\ell$  are connected, and no segment of  $M^\ell$  intersects  $\ell$ . Let  $\Pi_1$  and  $\Pi_2$  be the two halfplanes supported by  $\ell$ . Let  $P_i$  be the set of points of  $P$  that lie in  $\Pi_i$  and let  $M_i$  and  $H_i$  be, respectively, the set of segments of  $M^\ell$  and  $H$  that are contained in  $\Pi_i$ ,  $i \in \{1, 2\}$ .

Let  $\ell_1$  (resp.  $\ell_2$ ) be the ham-sandwich cut line of  $P_1$  (resp.  $P_2$ ) used to construct  $H$ . Solve the problem recursively for  $P_1, M_1, H_1$  and  $\ell_1$ , and for  $P_2, M_2, H_2$

and  $\ell_2$ . Since every  $BR$ -matching of  $P_1$  is compatible with every  $BR$ -matching of  $P_2$ , we can merge the two sequences obtained by the recursive construction that certify that  $M_i$  and  $H_i$  are connected,  $i \in \{1, 2\}$ . Thus,  $M^\ell$  is connected with  $H$  and since  $M$  is connected to  $M^\ell$ ,  $M$  and  $H$  are also connected.  $\square$

Let  $V$  be the set of all  $BR$ -matchings of  $P$  and let  $G_P$  be the graph with vertex set  $V$ , where there is an edge between two vertices if their corresponding matchings are compatible.

**Corollary 1.** *The graph  $G_P$  is connected.*

The following observation is depicted in Fig. 6(b).

**Observation 5** *There exist bichromatic point sets that admit  $BR$ -matchings at distance  $\Omega(n)$  in  $G_P$ .*

**Open questions** Future work could involve giving a tight bound on the diameter of  $G_P$ . It would also be interesting to bound the number of nodes in  $G_P$ .

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